Long-range order in lattice spin systems

2009 J. Phys. A: Math. Theor. 42 232002

(http://iopscience.iop.org/1751-8121/42/23/232002)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 232002 (5pp) doi:10.1088/1751-8113/42/23/232002

FAST TRACK COMMUNICATION

Long-range order in lattice spin systems

André F Verbeure

Institute for Theoretical Physics, K U Leuven, B-3001 Leuven-Heverlee, Belgium

E-mail: andre.verbeure@fys.kuleuven.be

Received 2 April 2009, in final form 7 May 2009
Published 22 May 2009
Online at stacks.iop.org/JPhysA/42/232002

Abstract
For equilibrium states of spin-reversal-invariant homogeneous classical spin systems, rigorous implications are shown from the notion of macroscopic occupation of the spin density to spontaneous spin symmetry breaking and long-range order, and vice versa.

PACS number: 05.20−y

The set of observables of a classical spin system on a d-dimensional lattice $\mathbb{Z}^d$ consists of the commutative algebra $\mathfrak{A}$ generated by the one-site observables $\{\sigma_x / x \in \mathbb{Z}^d\}$. In other words each system observable, denoted by X or $X(\sigma)$, is of the type

$$\sum_n \sum_{x_1,\ldots,x_n} c_{x_1,\ldots,x_n} \sigma_{x_1}\sigma_{x_2}\ldots\sigma_{x_n},$$

where $x_1, \ldots, x_n \in \mathbb{Z}^d$ and $c_{x_1,\ldots,x_n}$ are complex numbers. The spin variables σ_x take the values ± 1.

Homogeneous spin systems are defined by local Hamiltonians H_Λ, one for each finite subset Λ of the lattice, of the form

$$H_\Lambda = \Sigma_{\Delta\subseteq\Lambda}\phi(\Delta)\sigma_\Delta, \tag{1}$$

where we used the notation $\sigma_\Delta = \prod_{x\in\Delta}\sigma_x$. The translation invariance is guaranteed by the interaction energy condition $\phi(\Delta + a) = \phi(\Delta)$ holding for all lattice points a and subsets Δ of $\mathbb{Z}^d$.

The *global spin-flip operation* Θ maps each of the spin variables σ_x onto $-\sigma_x$. Also for Λ any finite subset of lattice points, we denote by Θ_Λ the *local spin-flip operation* of all spins σ_x with x in Λ.

Not only translation invariance of the systems is imposed, we also assume the *spin-flip invariance* of our systems, i.e. we assume that all local Hamiltonians satisfy the condition $\Theta(H_\Lambda) = H_\Lambda$ for all $\Lambda \subset \mathbb{Z}^d$.

Clearly, the best-known prototype model system is the d-dimensional Ising model: $H_\Lambda = -J\sum_{\langle x,y\rangle; x,y\in\Lambda}\sigma_x\sigma_y$, where $\langle x, y\rangle$ stand for the nearest neighbor sites x and y.

We are interested in the equilibrium states, which are expectation-valued maps or probability measures on the set of functions $\mathfrak{A}$ of these systems.

First, one can consider the limit *Gibbs states*, denoted by ω_β, with β being the inverse temperature, and naturally defined by the formula

$$\omega_\beta(X(\sigma)) = \lim_{\Lambda\to\mathbb{Z}^d} \frac{\sum_{\{\sigma=\pm1;\sigma\in\Lambda\}} X(\sigma)\exp\{-\beta H_\Lambda(\sigma)\}}{\sum_{\{\sigma=\pm1;\sigma\in\Lambda\}} \exp\{-\beta H_\Lambda(\sigma)\}}. \tag{2}$$

There are many possible thermodynamic limits $\Lambda \to \mathbb{Z}^d$, e.g. depending on the geometrical forms of Λ's, yielding several possibly different limit Gibbs states. By definition, any of these Gibbs states, denoted by the same symbol ω_β, is homogeneous and spin-flip invariant.

Any state ω of the spin system is an *equilibrium state*, if it satisfies the *energy-entropy balance (EEB) criterion at* β, i.e. if for each fixed finite lattice subset $\widetilde{\Lambda}$ and any non-negative observable $X \geqslant 0$, the following inequality holds:

$$\lim_{\Lambda} \omega(X(\sigma)(\Theta_{\widetilde{\Lambda}} H_\Lambda(\sigma) - H_\Lambda(\sigma))) \geqslant \frac{1}{\beta}\omega(X(\sigma)) \ln \frac{\omega(X(\sigma))}{\omega(\Theta_{\widetilde{\Lambda}}(X(\sigma)))}. \tag{3}$$

These conditions are handy tools as criteria for equilibrium states (prob. measures). In fact, they are nothing but the set of Euler equations for the basic free energy density functional variational principle of statistical mechanics of these systems. The reader should not be surprised by the inequalities instead of equalities, because it is proved that the inequalities are equivalent to the Euler equalities (see also [1]). The latter ones are however practically less manageable in the applications.

For all these reasons, the EEB criterion holds as firm general defining criteria for equilibrium states.

One shows [1] that each limit Gibbs state satisfies this EEB criterion. Each Gibbs state ω_β is clearly homogeneous and spin-flip invariant ($\omega_\beta \circ \Theta = \omega_\beta$). But there may exist more homogeneous states ω satisfying the EEB criterion. Some of them may break the Θ-symmetry invariance of the system $\{H_\Lambda\}$. If this happens, one speaks about the occurrence of *spontaneous symmetry breaking (SSB)*.

For any homogeneous state ω of the system, the *magnetization* of the state is given by

$$\omega(\sigma_y) = \lim_{\Lambda\to\mathbb{Z}^d} \omega\left(\frac{\sum_{x\in\Lambda}\sigma_x}{|\Lambda|}\right), \tag{4}$$

where y is any arbitrary lattice point and $|\Lambda|$ stands for the volume or the number of lattice points of Λ.

Moreover, the reader checks that averages of local observables (functions), say A, always exist (see the theorem of Kovacs–Szuecs in [2, chapter 4]) within the GNS representation [2] of a homogeneous state. For later use, the GNS representation is also called the Hilbert space representation of the set observables. The Hilbert space $\mathcal{H}$ is the closure of the set $\mathfrak{A}$ with respect to the scalar product $(X, Y) \equiv (X\Omega_\omega, Y\Omega_\omega) \equiv \omega(X^*Y)$, where $\Omega_\omega = \mathbf{1}$ stands for the the unit observable. With this in mind, if τ_a denotes the translation action $\tau_a(\sigma_x) = \sigma_{x+a}$ over distance a, then for all observables X, Y one has

$$\omega(X\overline{A}Y) = \lim_{\Lambda} \omega\left(X\left(\frac{1}{|\Lambda|}\sum_{a\in\Lambda}\tau_a A\right)Y\right) \tag{5}$$

defining the *average observable* $\overline{A} \equiv \lim_\Lambda \frac{1}{|\Lambda|}\sum_{a\in\Lambda}\tau_a A$.

Now take $A = \sigma_x$ for some point x; then $\overline{A} = \overline{\sigma_x} \equiv \overline{\sigma}$ and formula 4 becomes

$$\omega(\sigma_y) = \omega(\overline{\sigma}). \tag{6}$$

Now by Schwartz inequality, one gets

$$\omega(\sigma_y)^2 = \omega(\overline{\sigma})^2 \leqslant \omega(\overline{\sigma}^2) \tag{7}$$

expressing the following property. Namely, if ω is a homogeneous state breaking the symmetry, i.e. if $\omega(\sigma_y) \neq 0$, then $\omega(\overline{\sigma}^2) > 0$.

Clearly, any state ω with the property $\omega(\overline{\sigma}^2) > 0$ may be called in a natural sense a *state with macroscopic occupation of spin density*. In the rest of this communication, we show the following statement.

Statement. If one has a limit Gibbs state ω_β showing a macroscopic occupation of spin density, hence with the property $\omega_\beta(\overline{\sigma}^2) > 0$, then there exist spin symmetry breaking equilibrium (satisfying (3)) states ω_+ and ω_-, which satisfy the equality

$$\omega(\sigma_y)^2_\pm = \omega_\pm(\overline{\sigma})^2 = \omega_\pm(\overline{\sigma}^2), \tag{8}$$

i.e. they satisfy formula (7) but with the equality sign. Also, the Gibbs state is an equal weight convex combination of the states $\omega_\pm$.

The implications of this equality sign case are as follows. If one has spontaneous symmetry breaking states $\omega_\pm$ and the equality sign in (8), then these states have the property of showing the long-range order property

$$\omega_\pm(\overline{\sigma}^2) = |\omega_\pm(\overline{\sigma})|^2 > 0. \tag{9}$$

This property is similar to the notion of 'off-diagonal long-range order' (see [3, 4]), a notion which has been introduced in the context of quantized fields.

The statement also expresses that, if one has a Gibbs state ω_β showing a macroscopic occupation of spin density, then the symmetry breaking states $\omega_\pm$ always exist and they show long-range order.

Construction of the states $\omega_\pm$. We start from the given limit Gibbs state ω_β which is homogeneous, spin-flip invariant and satisfies the property $\omega_\beta(\overline{\sigma}^2) > 0$. In particular, spin-flip invariance implies that $\omega_\beta(\sigma_{x_1} ... \sigma_{x_{2n+1}}) = 0$ for all integers n. Consider the average spin function $\overline{\sigma}$ (5) in the representation induced by the state ω_β. This average is a real function with values in the interval $[-1, 1]$. Consider the polar decomposition (see e.g. [5]) of this average function

$$\overline{\sigma} = U\sqrt{\overline{\sigma}^2}. \tag{10}$$

As $\omega_\beta(\overline{\sigma}^2) > 0$, one has $\overline{\sigma} \neq 0$ or $\overline{\sigma}$ is a non-trivial function in this representation. U is a real function taking the values ± 1 or $U^2 = 1$ on the support of $\overline{\sigma}$. One can also write $U = \overline{\sigma}/\sqrt{\overline{\sigma}^2}$. U can also be extended by 1 outside the support, such that $U^2 = 1$ everywhere.

Define the new spin variables for all x in the lattice

$$\widetilde{\sigma}_x = U\sigma_x \equiv \eta(\sigma_x),$$

where η is a morphism of the algebra $\mathfrak{A}$ generated by $\sigma's$ into itself. The new variables $\widetilde{\sigma}_x$ generate a new representation of the original observables. Now define the states $\omega_\pm$ as follows: for each observable X,

$$\omega_+(X) = \omega_\beta(\eta(X)), \omega_-(X) = \omega_\beta(\eta(\Theta(X))). \tag{11}$$

Properties of the states $\omega_\pm$.

(i) It is readily checked, using the definition formulae 11 based on the given Gibbs state ω_β, with β finite, that the states $\omega_\pm$ satisfy the EEB criterion 3. Therefore, the newly constructed states are equilibrium states.

(ii) One computes that

$$\omega_\pm(\sigma_{x_1} \cdots \sigma_{x_{2n}}) = \omega_\beta(\sigma_{x_1} \cdots \sigma_{x_{2n}})$$

implying that the new states and the Gibbs state coincide on the even monomials in σ's. Also, as $U^2 = 1$,

$$\omega_+(\sigma_{x_1}\cdots\sigma_{x_{2n+1}}) = -\omega_-(\sigma_{x_1}\cdots\sigma_{x_{2n+1}})$$

implying altogether that for all observables X holds:

$$\omega_\beta(X) = \tfrac{1}{2}\omega_+(X) + \tfrac{1}{2}\omega_-(X). \tag{12}$$

The given Gibbs state ω_β is written as an equal weight convex combination of the two constructed states $\omega_\pm$.

(iii) From the definition formulae 11, one again computes the formula

$$\omega_\pm(\overline{\sigma}) = \pm\omega_\beta(\sqrt{\overline{\sigma}^2}). \tag{13}$$

Using now the Hilbert space representations (GNS representation [2]) of the states $\omega_\pm$, respectively ω_β:

$$\omega_\pm(X) = (\Omega_\pm, X\Omega_\pm);\ \omega_\beta(X) = (\Omega_\beta, X\Omega_\beta).$$

Then

$$\omega_\beta(\sqrt{\overline{\sigma}^2})^2 = |\omega_\pm(\overline{\sigma})|^2 = |(\Omega_\pm, U\sqrt{\overline{\sigma}^2}\Omega_\pm)|^2 = |(\sqrt[4]{\overline{\sigma}^2}\Omega_\pm, U\sqrt[4]{\overline{\sigma}^2}\Omega_\pm)|^2,$$

which by Schwartz inequality is majorized by

$$\leqslant (\Omega_\pm, \sqrt{\overline{\sigma}^2}\Omega_\pm)^2 = \omega_\beta(\sqrt{\overline{\sigma}^2})^2.$$

This implies that the vector $\Omega_\pm$ is proportional to the vector $U\sqrt{\overline{\sigma}^2}\Omega_\pm$ or that there exists a complex number κ such that

$$\kappa\Omega_\pm = U\sqrt{\overline{\sigma}^2}\Omega_\pm = \overline{\sigma}\Omega_\pm.$$

One gets

$$|\omega_\pm(\overline{\sigma})|^2 = |(\Omega_\pm, \overline{\sigma}\Omega_\pm)|^2 = |\kappa|^2 = (\overline{\sigma}\Omega_\pm, \overline{\sigma}\Omega_\pm) = \omega_\pm(\overline{\sigma}^2) = \omega_\beta(\overline{\sigma}^2) > 0.$$

The last inequality follows from the property of macroscopic occupation of spin density for the state ω_β. This relation proves that the states $\omega_\pm$ have the property of showing SSB as well as that of showing long-range order (8).

All this proves the statement.

Remarks. The reader commented on the canonical-model-independent construction (11) of the SSB states. He surely remembers the usual ways of showing the existence of spontaneous symmetry breaking states for the Ising systems, which consist of fixing plus or minus symmetry breaking boundary conditions. There are also the plus combined minus boundary conditions leading to non-homogeneous states with an interface structure. Non-homogeneous states are outside the scope of this communication. In [6], one finds the main results for Ising systems. The $\pm$ boundary condition technique is considered to give a physical interpretation of the origins of the symmetry breaking. On the other hand, one can also consider external field perturbations, which tend to zero at the end of the argument. Model computations [7] show boundary condition dependences on the various volume rates at which this limit is taken to be zero. In any case not only for quantum systems but also for classical systems, boundary conditions can show a complicated picture (see e.g. [8]). The construction (11) of the SSB states is boundary conditions independent. It is based on the notion of macroscopic occupation of spin density for a limit Gibbs state.

Finally, we stress that our construction of SSB states yields an immediate and explicit relation between a limit Gibbs state (ω_β) and the SSB states. For convenience, we rewrite the SSB states once more in the following form: for any $n \in \mathbb{N}$, one has

$$\omega_+(\sigma_{x_1} \cdots \sigma_{x_n}) = \omega_\beta \left(\left(\frac{\overline{\sigma}}{\sqrt{\overline{\sigma}^2}} \right)^n \sigma_{x_1} \cdots \sigma_{x_n} \right)$$

$$\omega_-(\sigma_{x_1} \cdots \sigma_{x_n}) = (-1)^n \omega_\beta \left(\left(\frac{\overline{\sigma}}{\sqrt{\overline{\sigma}^2}} \right)^n \sigma_{x_1} \cdots \sigma_{x_n} \right).$$

Acknowledgments

It is a pleasure to thank Mark Fannes for discussions on the subject matter.

References

[1] Fannes M, Vanheuverzwijn P and Verbeure A 1982 Energy entropy inequalities for classical lattice systems *J. Stat. Phys.* **29** 547–60
[2] Bratteli O and Robinson D W 1979 *Operator Algebras and Statistical Mechanics, Part I* 1st edn (Berlin: Springer)
Bratteli O and Robinson D W 1996 *Operator Algebras and Statistical Mechanics, Part II* 2nd edn (Berlin: Springer)
[3] Penrose O and Onsager L 1956 Bose–Einstein condensation and liquid helium *Phys. Rev.* **104** 576
[4] Yang C N 1962 *Rev. Mod. Phys.* **34** 838
[5] Dunford N and Schwartz J T 1963 *Linear Operators, Part II* (New York: Interscience)
[6] Simon B 1993 *The Statistical Mechanics of Lattice Gases* vol 1 (Princeton, NJ: Princeton University Press)
[7] Verbeure A F and Zagrebnov V A 1994 Gaussian, non-Gaussian critical fluctuations in the Curie–Weiss model *J. Stat. Phys.* **75** 1137–52
[8] Fannes M and Werner R F 1995 Boundary conditions for quantum lattice systems *Helv. Phys. Acta* **68** 635–57